Limits – Indeterminate Forms and L'Hospital's Rule

I. Indeterminate Form of the Type $\frac{0}{0}$

We have previously studied limits with the indeterminate form $\frac{0}{0}$ as shown in the following examples:

Example 1: $\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x+2)\cancel{(x-2)}}{\cancel{x-2}}=\lim_{x\to 2}(x+2)=2+2=4$

Example 2: $\lim_{x\to 0}\frac{\tan 3x}{\sin 2x}=\lim_{x\to 0}\frac{\frac{\sin 3x}{\cos 3x}}{\sin 2x}=\lim_{x\to 0}\frac{\sin 3x}{1}\cdot\frac{1}{\cos 3x}\cdot\frac{1}{\sin 2x}=$

$$\frac{3}{2}\left(\lim_{3x\to 0}\frac{\sin 3x}{3x}\right)\left(\lim_{x\to 0}\frac{1}{\cos 3x}\right)\left(\lim_{2x\to 0}\frac{2x}{\sin 2x}\right)=\frac{3}{2}(1)(1)(1)=\frac{3}{2}$$

[*Note*: We use the given limit $\lim_{\Delta\to 0}\frac{\sin\Delta}{\Delta}=1$.]

Example 3: $\lim_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}=f'(8)=\frac{1}{3\sqrt[3]{8^2}}=\frac{1}{12}$. [*Note*: We use the definition of the derivative $f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ where $f(x)=\sqrt[3]{x}$ and a = 8.]

Example 4: $\lim_{x\to \pi/3}\frac{\cos x-1/2}{x-\pi/3}=f'(\pi/3)=-\sin(\pi/3)=-\sqrt{3}/2$. [*Note*: We use the definition of the derivative $f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ where $f(x)=\cos x$ and $a=\pi/3$.]

However, there is a general, systematic method for determining limits with the indeterminate form $\frac{0}{0}$. Suppose that *f* and *g* are differentiable functions at $x = a$ and that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of the type $\frac{0}{0}$; that is, $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Since *f* and *g* are differentiable functions at $x = a$, then *f* and *g* are continuous at $x = a$; that is, $f(a) = \lim_{x \to a} f(x) = 0$ and $g(a) = \lim_{x \to a} g(x) = 0$. Furthermore, since *f* and *g* are differentiable functions at $x = a$, then $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ and $g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$. Thus, if $g'(a) \neq 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{\frac{f(x) - f(a)}{x - a}}{\frac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if f' and g' are continuous at $x = a$. This illustrates a special case of the technique known as **L'Hospital's Rule**.

L'Hospital's Rule for Form $\frac{0}{0}$

Suppose that *f* and *g* are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty$, or as $x \to +\infty$.

In the following examples, we will use the following three-step process:

Step 1. Check that the limit of $\frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$**.** If it is not, then **L'Hospital's Rule** cannot be used.

Step 2. Differentiate *f* and *g* separately. [*Note*: **Do not differentiate $\frac{f(x)}{g(x)}$ using the quotient rule!**]

Step 3. Find the limit of $\frac{f'(x)}{g'(x)}$. If this limit is finite, $+\infty$, or $-\infty$, then it is equal to the limit of $\frac{f(x)}{g(x)}$**.** If the limit is an indeterminate form of type $\frac{0}{0}$**,** then simplify $\frac{f'(x)}{g'(x)}$ algebraically and apply **L'Hospital's Rule** again.

Example 1: $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}=\lim\limits_{x\to 2}\frac{2x}{1}=2(2)=4$

Example 2: $\lim\limits_{x\to 0}\frac{\tan 3x}{\sin 2x}=\lim\limits_{x\to 0}\frac{3\sec^2 3x}{2\cos 2x}=\frac{3(1)}{2(1)}=\frac{3}{2}$

Example 3: $\lim\limits_{h\to 0}\frac{\sqrt[3]{8+h}-2}{h}=\lim\limits_{h\to 0}\frac{\frac{1}{3}(8+h)^{-2/3}(1)}{1}=\lim\limits_{h\to 0}\frac{1}{3(8+h)^{2/3}}=\frac{1}{3(8)^{2/3}}=\frac{1}{12}$

Example 4: $\lim\limits_{x\to\pi/3}\frac{\cos x-1/2}{x-\pi/3}=\lim\limits_{x\to\pi/3}\frac{-\sin x}{1}=-\sin\left(\pi/3\right)=-\frac{\sqrt{3}}{2}$

Example 5: $\lim\limits_{x\to 0}\frac{e^x-x-1}{x^2}=\lim\limits_{x\to 0}\frac{e^x-1}{2x}=\lim\limits_{x\to 0}\frac{e^x}{2}=\frac{1}{2}$ [Use **L'Hospital's Rule** twice.]

Example 6: $\lim\limits_{x\to+\infty}\frac{1/x^2}{\sin\left(1/x\right)}=\lim\limits_{x\to+\infty}\frac{-2/x^3}{\cos\left(1/x\right)\left(-1/x^2\right)}=\lim\limits_{x\to+\infty}\frac{2/x}{\cos\left(1/x\right)}=\frac{0}{1}=0$, or

$\lim\limits_{x\to+\infty}\frac{1/x^2}{\sin\left(1/x\right)}=\lim\limits_{y\to 0^+}\frac{y^2}{\sin y}=\lim\limits_{y\to 0^+}\frac{2y}{\cos y}=\frac{2(0)}{1}=0$ where $y=1/x$.

Example 7: $\lim\limits_{x\to 0}\frac{x}{\ln x}=\lim\limits_{x\to 0}x\left(1/\ln x\right)=0(0)=0$ [This limit is **not** an indeterminate form of the type $\frac{0}{0}$, so **L'Hospital's Rule** cannot be used.]

II. Indeterminate Form of the Type $\frac{\infty}{\infty}$

We have previously studied limits with the indeterminate form $\frac{\infty}{\infty}$ as shown in the following examples:

Example 1: $$\lim_{x\to+\infty}\frac{3x^2+5x-7}{2x^2-3x+1}=\lim_{x\to+\infty}\frac{\frac{3x^2}{x^2}+\frac{5x}{x^2}-\frac{7}{x^2}}{\frac{2x^2}{x^2}-\frac{3x}{x^2}+\frac{1}{x^2}}=$$

$$\lim_{x\to+\infty}\frac{3+\frac{5}{x}-\frac{7}{x^2}}{2-\frac{3}{x}+\frac{1}{x^2}}=\lim_{x\to+\infty}\frac{3+0-0}{2-0+0}=\frac{3}{2}$$

Example 2: $$\lim_{x\to-\infty}\frac{3x-1}{x^2+1}=\lim_{x\to-\infty}\frac{\frac{3x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to-\infty}\frac{\frac{3}{x}-\frac{1}{x^2}}{1+\frac{1}{x^2}}=\frac{0-0}{1+0}=\frac{0}{1}=0$$

Example 3: $$\lim_{x\to\infty}\frac{3x^3-4}{2x^2+1}=\lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4}{x^3}}{\frac{2x^2}{x^3}+\frac{1}{x^3}}=\lim_{x\to\infty}\frac{3-\frac{4}{x^3}}{\frac{2}{x}+\frac{1}{x^3}}=\frac{3-0}{0+0}=\frac{3}{0}\Rightarrow$$

limit does not exist.

Example 4: $$\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}}{x+1}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+1}}{x}}{\frac{x+1}{x}}=\lim_{x\to-\infty}\frac{\frac{\sqrt{4x^2+1}}{-\sqrt{x^2}}}{\frac{x+1}{x}}$$ ($\sqrt{x^2}=-x$

because $x<0$ and thus $x=-\sqrt{x^2}$) = $$\lim_{x\to-\infty}\frac{-\sqrt{\frac{4x^2+1}{x^2}}}{\frac{x+1}{x}}=$$

$$\lim_{x\to-\infty} \frac{-\sqrt{4+\dfrac{1}{x^2}}}{1+\dfrac{1}{x^2}} = \frac{-\sqrt{4}}{1} = -2.$$

However, we could use another version of **L'Hospital's Rule**.

L'Hospital's Rule for Form $\dfrac{\infty}{\infty}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. If $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ has a finite limit, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} \dfrac{f'(x)}{g'(x)}$. Moreover, this statement is also true in the case of a limit as $x \to a^-, x \to a^+, x \to -\infty,$ or as $x \to +\infty$.

Example 1: $\displaystyle\lim_{x\to+\infty} \frac{3x^2+5x-7}{2x^2-3x+1} = \lim_{x\to+\infty} \frac{6x+5}{4x-3} = \lim_{x\to+\infty} \frac{6}{4} = \frac{3}{2}$

Example 2: $\displaystyle\lim_{x\to-\infty} \frac{3x-1}{x^2+1} = \lim_{x\to-\infty} \frac{3}{2x} = \frac{3}{2}\lim_{x\to-\infty} \frac{1}{x} = \frac{3}{2}(0) = 0$

Example 3: $\displaystyle\lim_{x\to\infty} \frac{3x^3-4}{2x^2+1} = \lim_{x\to\infty} \frac{9x^2}{4x} = \lim_{x\to\infty} \frac{18x}{4} = \infty$

Example 4: $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{4x^2+1}}{x+1} = \lim_{x\to-\infty} \frac{\dfrac{8x}{2\sqrt{4x^2+1}}}{1} = \lim_{x\to-\infty} \frac{4x}{\sqrt{4x^2+1}}$ $\Rightarrow$ **L'Hospital's Rule** does not help in this situation. We would find the limit as we did previously.

Example 5: $\lim_{x\to+\infty} \frac{\ln(x^2+1)}{\ln(x^3+1)} = \lim_{x\to+\infty} \frac{\frac{2x}{x^2+1}}{\frac{3x^2}{x^3+1}} = \lim_{x\to+\infty} \frac{2x(x^3+1)}{3x^2(x^2+1)} = \lim_{x\to+\infty} \frac{2x^4+2x}{3x^4+3x^2} =$

$\lim_{x\to+\infty} \frac{8x^3+2}{12x^3+6x} = \lim_{x\to+\infty} \frac{24x^2}{36x^2+6} = \lim_{x\to+\infty} \frac{48x}{72x} = \frac{48}{72} = \frac{2}{3}$

Example 6: $\lim_{x\to0^+} \frac{\ln x}{1/x^2} = \lim_{x\to0^+} \frac{1/x}{-2/x^3} = \lim_{x\to0^+} \frac{x^3}{-2x} = \lim_{x\to0^+} \frac{x^2}{-2} = \frac{0^2}{-2} = 0$

Example 7: $\lim_{x\to+\infty} \frac{\arctan x}{x} = \left(\lim_{x\to+\infty} \frac{1}{x}\right)\left(\lim_{x\to+\infty} \arctan x\right) = (0)\left(\frac{\pi}{2}\right) = 0$ [This limit is **not** an indeterminate form of the type $\frac{\infty}{\infty}$, so **L'Hospital's Rule** cannot be used.]

III. Indeterminate Form of the Type $0 \cdot \infty$

Indeterminate forms of the type $0 \cdot \infty$ can sometimes be evaluated by rewriting the product as a quotient, and then applying **L'Hospital's Rule** for the indeterminate forms of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 1: $\lim_{x\to0^+} x\ln x = \lim_{x\to0^+} \frac{\ln x}{1/x} = \lim_{x\to0^+} \frac{1/x}{-1/x^2} = \lim_{x\to0^+} \frac{-x^2}{x} = \lim_{x\to0^+} (-x) = 0$

Example 2: $\lim_{x\to0^+} (\sin x)\ln x = \lim_{x\to0^+} \frac{\ln x}{\csc x} = \lim_{x\to0^+} \frac{1/x}{-\csc x \cot x} = \lim_{x\to0^+} \frac{-\sin x \tan x}{x} =$

$\left(\lim_{x\to0^+} \frac{-\sin x}{x}\right)\left(\lim_{x\to0^+} \tan x\right) = (-1)(0) = 0$

Example 3: $\lim_{x\to+\infty} x\sin\left(1/x\right) = \lim_{x\to+\infty} \frac{\sin\left(1/x\right)}{1/x} = \lim_{y\to0^+} \frac{\sin y}{y} = 1$ [Let $y = 1/x$.]

IV. Indeterminate Form of the Type $\infty - \infty$

A limit problem that leads to one of the expressions

$$(+\infty)-(+\infty),\quad (-\infty)-(-\infty),\quad (+\infty)+(-\infty),\quad (-\infty)+(+\infty)$$

is called an **indeterminate form of type** $\infty - \infty$. Such limits are indeterminate because the two terms exert conflicting influences on the expression; one pushes it in the positive direction and the other pushes it in the negative direction. However, limits problems that lead to one the expressions

$$(+\infty)+(+\infty),\quad (+\infty)-(-\infty),\quad (-\infty)+(-\infty),\quad (-\infty)-(+\infty)$$

are not indeterminate, since the two terms work together (the first two produce a limit of $+\infty$ and the last two produce a limit of $-\infty$). Indeterminate forms of the type $\infty - \infty$ can sometimes be evaluated by combining the terms and manipulating the result to produce an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example 1: $\lim\limits_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{\sin x}\right)=\lim\limits_{x\to 0^+}\left(\frac{\sin x - x}{x\sin x}\right)=\lim\limits_{x\to 0^+}\frac{\cos x - 1}{x\cos x + \sin x}=$

$\lim\limits_{x\to 0^+}\frac{-\sin x}{-x\sin x+\cos x+\cos x}=\frac{0}{2}=0$

Example 2: $\lim\limits_{x\to 0}\left[\ln(1-\cos x)-\ln\left(x^2\right)\right]=\lim\limits_{x\to 0}\left[\ln\left(\frac{1-\cos x}{x^2}\right)\right]=$

$\ln\left[\lim\limits_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\right]=\ln\left[\lim\limits_{x\to 0}\left(\frac{\sin x}{2x}\right)\right]=\ln\left(\frac{1}{2}\right)$

V. Indeterminate Forms of the Types 0^0, ∞^0, 1^∞

Limits of the form $\lim\limits_{x\to a}[f(x)]^{g(x)}$ $\left\{or\ \lim\limits_{x\to\infty}[f(x)]^{g(x)}\right\}$ frequently give rise to indeterminate forms of the types 0^0, ∞^0, 1^∞. These indeterminate forms can sometimes be evaluated as follows:

(1) $y=[f(x)]^{g(x)}$
(2) $\ln y=\ln[f(x)]^{g(x)} = g(x)\ln[f(x)]$
(3) $\lim\limits_{x\to a}[\ln y]=\lim\limits_{x\to a}\{g(x)\ln[f(x)]\}$

The limit on the righthand side of the equation will usually be an indeterminate limit of the type $0 \cdot \infty$. Evaluate this limit using the technique previously described. Assume that $\lim_{x \to a} \{g(x) \ln[f(x)]\} = L.$

(4) Finally, $\lim_{x \to a} [\ln y] = L \Rightarrow \ln\left[\lim_{x \to a} y\right] = L \Rightarrow \lim_{x \to a} y = e^L$.

Example 1: Find $\lim_{x \to 0^+} x^x$.

This is an indeterminate form of the type 0^0. Let $y = x^x \Rightarrow \ln y = \ln x^x =$ $x \ln x$. $\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{1/x}{-1/x^2} = \lim_{x \to 0^+} (-x) = 0.$

Thus, $\lim_{x \to 0^+} x^x = e^0 = 1$.

Example 2: Find $\lim_{x \to +\infty} (e^x + 1)^{-2/x}$.

This is an indeterminate form of the type ∞^0. Let $y = (e^x + 1)^{-2/x} \Rightarrow$ $\ln y = \ln\left[(e^x + 1)^{-2/x}\right] = \frac{-2\ln(e^x + 1)}{x}$. $\lim_{x \to +\infty} \ln y = \lim_{x \to +\infty} \frac{-2\ln(e^x + 1)}{x} =$ $\lim_{x \to +\infty} \frac{-2\left(\frac{e^x}{e^x + 1}\right)}{1} = \lim_{x \to +\infty} \frac{-2e^x}{e^x + 1} = \lim_{x \to +\infty} \frac{-2e^x}{e^x} = -2$. Thus, $\lim_{x \to +\infty} (e^x + 1)^{-2/x} =$ e^{-2}.

Example 3: Find $\lim_{x \to 0^+} (\cos x)^{1/x}$.

This is an indeterminate form of the type 1^∞. Let $y = (\cos x)^{1/x} \Rightarrow$ $\ln y = \ln\left[(\cos x)^{1/x}\right] = \frac{\ln(\cos x)}{x}$. $\lim_{x \to 0^+} \ln y = \lim_{x \to 0^+} \frac{\ln(\cos x)}{x} =$ $\lim_{x \to 0^+} (-\tan x) = 0$. Thus, $\lim_{x \to 0^+} (\cos x)^{1/x} = e^0 = 1$.

Practice Sheet for L'Hospital's Rule

(1) $\lim\limits_{x\to 0} \dfrac{xe^{3x} - x}{1-\cos(2x)} =$

(2) $\lim\limits_{x\to +\infty} \dfrac{x}{(\ln x)^3} =$

(3) $\lim\limits_{x\to 0} [\ln(1-\cos x) - \ln(x^2)] =$

(4) $\lim\limits_{x\to +\infty} \left(1-\dfrac{2}{x}\right)^{3x} =$

(5) $\lim\limits_{x\to +\infty} \dfrac{\cos\left(\frac{1}{x}\right)-1}{\cos\left(\frac{2}{x}\right)-1} =$

(6) $\lim\limits_{x\to 0} \dfrac{\sqrt{1-x}-\sqrt{1-x^2}}{x} =$

(7) $\lim\limits_{x\to 0} (\cos x)^{1/x^2} =$

(8) $\lim\limits_{x\to 1} \dfrac{5x^4 - 7x^3 + x^2 - x + 2}{3x^4 - 8x^3 + 6x^2 - 1} =$

(9) $\lim\limits_{x\to 0} \dfrac{9-\sqrt{81-5x}}{x} =$

(10) $\lim\limits_{x\to +\infty} \dfrac{\ln\left(x^3+2\right)}{\ln\left(5x^3-1\right)} =$

(11) $\lim\limits_{x\to +\infty} \left(e^x+1\right)^{-2/x} =$

(12) $\lim_{x\to 0} \frac{\sin(4x) - 2\sin(2x)}{x^3} =$

(13) $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x} \right] =$

(14) $\lim_{x\to +\infty} x\left(e^{1/x} - 1 \right) =$

(15) $\lim_{x\to 0^+} \sqrt[3]{x} \ln x =$

(16) $\lim_{x\to 0} \frac{\ln\left(\frac{2x+1}{5x+1} \right)}{x} =$

(17) $\lim_{x\to +\infty} \left(1 + \frac{e}{x} \right)^{x/2} =$

(18) $\lim_{x\to 0} \frac{\arctan(\sin 3x)}{x} =$

(19) $\lim_{x\to 0^+} \frac{\int_0^x \sin\left(t^2\right) dt}{x^3} =$

(20) $\lim_{x\to 0} \left(e^{2x} + x \right)^{1/x} =$

(21) $\lim_{x\to +\infty} \frac{\arctan x}{x} =$

(22) $\lim_{x\to 0} \frac{\arctan(\sin(3x))}{\arcsin(2\tan x)} =$

(23) $\lim_{x\to 0} \frac{\ln(\cos x)}{x^2} =$

(24) $\lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x} =$

(25) $\lim_{x\to+\infty}(\ln x)^{1/x} =$

(26) $\lim_{x\to+\infty}\left(\ln(e^{x}+1)-x\right)=$

Solution Key for L'Hospital's Rule

(1) $\lim_{x\to 0}\frac{xe^{3x}-x}{1-\cos(2x)} = \lim_{x\to 0}\frac{3xe^{3x}+e^{3x}-1}{2\sin 2x} = \lim_{x\to 0}\frac{9xe^{3x}+3e^{3x}+3e^{3x}}{4\cos 2x} = \frac{6}{4} = \frac{3}{2}$

(2) $\lim_{x\to+\infty}\frac{x}{(\ln x)^3} = \lim_{x\to+\infty}\frac{1}{3(\ln x)^2\left(1/x\right)} = \lim_{x\to+\infty}\frac{x}{3(\ln x)^2} = \lim_{x\to+\infty}\frac{1}{6\ln x\left(1/x\right)} = \lim_{x\to+\infty}\frac{x}{6\ln x} =$

$\lim_{x\to+\infty}\frac{1}{6\left(1/x\right)} = \lim_{x\to+\infty}\frac{x}{6} = +\infty$

(3) $\lim_{x\to 0}[\ln(1-\cos x)-\ln(x^2)] = \lim_{x\to 0}\ln\left(\frac{1-\cos x}{x^2}\right) = \ln\left\{\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right)\right\} =$

$\ln\left\{\lim_{x\to 0}\left(\frac{\sin x}{2x}\right)\right\} = \ln\left(\frac{1}{2}\right) = -\ln 2$

(4) Let $y=\frac{1}{x} \Rightarrow \lim_{x\to+\infty}\left(1-\frac{2}{x}\right)^{3x} = \lim_{y\to 0^+}(1-2y)^{3/y}$. Now, let $z=(1-2y)^{3/y} \Rightarrow \ln z =$

$\ln(1-2y)^{3/y} = \frac{3\ln(1-2y)}{y} \Rightarrow \lim_{y\to 0^+}\ln z = \lim_{y\to 0^+}\frac{3\ln(1-2y)}{y} = \lim_{y\to 0^+}\frac{\frac{-6}{1-2y}}{1} = -6$. Thus,

$\lim_{y\to 0^+}\ln z = -6 \Rightarrow \ln\left(\lim_{y\to 0^+} z\right) = -6 \Rightarrow \lim_{y\to 0^+} z = e^{-6} \Rightarrow \lim_{x\to+\infty}\left(1-\frac{2}{x}\right)^{3x} = \lim_{y\to 0^+}(1-2y)^{3/y} =$

$\lim_{y\to 0^+} z = e^{-6}$.

(5) Let $y = \frac{1}{x} \Rightarrow \lim_{x\to+\infty} \frac{\cos\left(\frac{1}{x}\right)-1}{\cos\left(\frac{2}{x}\right)-1} = \lim_{y\to0^+} \frac{\cos(y)-1}{\cos(2y)-1} = \lim_{y\to0^+} \frac{-\sin(y)}{-2\sin(2y)} =$

$\lim_{y\to0^+} \frac{\sin y}{4\sin y\cos y} = \lim_{y\to0^+} \frac{1}{4\cos y} = \frac{1}{4}$.

(6) $\lim_{x\to0} \frac{\sqrt{1-x}-\sqrt{1-x^2}}{x} = \lim_{x\to0} \frac{\sqrt{1-x}-\sqrt{1-x^2}}{x} \cdot \frac{\sqrt{1-x}+\sqrt{1-x^2}}{\sqrt{1-x}+\sqrt{1-x^2}} =$

$\lim_{x\to0} \frac{(1-x)-(1-x^2)}{x\left(\sqrt{1-x}+\sqrt{1-x^2}\right)} = \lim_{x\to0} \frac{x^2-x}{x\left(\sqrt{1-x}+\sqrt{1-x^2}\right)} = \lim_{x\to0} \frac{x-1}{\sqrt{1-x}+\sqrt{1-x^2}} = -\frac{1}{2}$.

(7) Let $y = (\cos x)^{1/x^2} \Rightarrow \ln y = \ln(\cos x)^{1/x^2} = \frac{\ln(\cos x)}{x^2} \Rightarrow \lim_{x\to0} (\ln y) = \lim_{x\to0} \frac{\ln(\cos x)}{x^2} =$

$\lim_{x\to0} \frac{\frac{-\sin x}{\cos x}}{2x} = \lim_{x\to0} \frac{-\sin x}{2x\cos x} = \lim_{x\to0} \left(\frac{\sin x}{x}\right)\left(\frac{-1}{2\cos x}\right) = -\frac{1}{2}$. Thus, $\lim_{x\to0} (\ln y) = -\frac{1}{2} \Rightarrow$

$\ln\left(\lim_{x\to0} y\right) = -\frac{1}{2} \Rightarrow \lim_{x\to0} y = e^{-1/2} \Rightarrow \lim_{x\to0} (\cos x)^{1/x^2} = \lim_{x\to0} y = e^{-1/2}$.

(8) $\lim_{x\to1} \frac{5x^4-7x^3+x^2-x+2}{3x^4-8x^3+6x^2-1} = \lim_{x\to1} \frac{20x^3-21x^2+2x-1}{12x^3-24x^2+12x} = \lim_{x\to1} \frac{60x^2-42x+2}{36x^2-48x+12} =$

$\frac{20}{0} \Rightarrow$ limit does not exist.

(9) $\lim_{x\to0} \frac{9-\sqrt{81-5x}}{x} = \lim_{x\to0} \frac{9-\sqrt{81-5x}}{x} \cdot \frac{9+\sqrt{81-5x}}{9+\sqrt{81-5x}} = \lim_{x\to0} \frac{81-(81-5x)}{x\left(9+\sqrt{81-5x}\right)} =$

$\lim_{x\to0} \frac{5x}{x\left(9+\sqrt{81-5x}\right)} = \lim_{x\to0} \frac{5}{9+\sqrt{81-5x}} = \frac{5}{18}$.

(10) $\lim_{x\to+\infty} \frac{\ln\left(x^3+2\right)}{\ln\left(5x^3-1\right)} = \lim_{x\to+\infty} \frac{\frac{3x^2}{x^3+2}}{\frac{15x^2}{5x^3-1}} = \lim_{x\to+\infty} \frac{3(5x^3-1)}{15(x^3+2)} = \lim_{x\to+\infty} \frac{15x^3-3}{15x^3+30} = 1$

(11) Let $y=\left(e^x+1\right)^{-2/x}\Rightarrow \ln y=\ln\left(e^x+1\right)^{-2/x}=\dfrac{-2\ln(e^x+1)}{x}\Rightarrow \lim\limits_{x\to+\infty}\ln y=$

$$\lim_{x\to+\infty}\frac{-2\ln(e^x+1)}{x}=\lim_{x\to+\infty}\frac{\frac{-2e^x}{e^x+1}}{1}=\lim_{x\to+\infty}\frac{-2e^x}{e^x+1}=\lim_{x\to+\infty}\frac{-2e^x}{e^x}=-2.$$ Thus, $\lim\limits_{x\to+\infty}\ln y=$

$$-2\Rightarrow \ln\left(\lim_{x\to+\infty}y\right)=-2\Rightarrow \lim_{x\to+\infty}y=e^{-2}\Rightarrow \lim_{x\to+\infty}\left(e^x+1\right)^{-2/x}=\lim_{x\to+\infty}y=e^{-2}.$$

(12) $$\lim_{x\to0}\frac{\sin(4x)-2\sin(2x)}{x^3}=\lim_{x\to0}\frac{4\cos(4x)-4\cos(2x)}{3x^2}=\lim_{x\to0}\frac{-16\sin(4x)+8\sin(2x)}{6x}=$$

$$\lim_{x\to0}\frac{-64\cos(4x)+16\cos(2x)}{6}=\frac{-48}{6}=-8.$$

(13) $$\lim_{x\to0}\left[\frac{1}{\sin x}-\frac{1}{x}\right]=\lim_{x\to0}\left(\frac{x-\sin x}{x\sin x}\right)=\lim_{x\to0}\frac{1-\cos x}{x\cos x+\sin x}=$$

$$\lim_{x\to0}\frac{\sin x}{-x\sin x+\cos x+\cos x}=\frac{0}{2}=0.$$

(14) $$\lim_{x\to+\infty}x\left(e^{1/x}-1\right)=\lim_{x\to+\infty}\frac{e^{1/x}-1}{1/x}=\lim_{x\to+\infty}\frac{e^{1/x}\left(-1/x^2\right)}{-1/x^2}=\lim_{x\to+\infty}e^{1/x}=e^0=1.$$

(15) $$\lim_{x\to0^+}\sqrt[3]{x}\ln x=\lim_{x\to0^+}\frac{\ln x}{x^{-1/3}}=\lim_{x\to0^+}\frac{1/x}{-\frac{1}{3}x^{-4/3}}=\lim_{x\to0^+}\frac{-3x^{4/3}}{x}=\lim_{x\to0^+}(-3\sqrt[3]{x})=0.$$

(16) $$\lim_{x\to0}\frac{\ln\left(\frac{2x+1}{5x+1}\right)}{x}=\lim_{x\to0}\frac{\left(\frac{5x+1}{2x+1}\right)\left(\frac{2(5x+1)-5(2x+1)}{(5x+1)^2}\right)}{1}=\lim_{x\to0}\frac{-3}{(2x+1)(5x+1)}=-3.$$

(17) Let $y=\dfrac{1}{x}\Rightarrow \lim\limits_{x\to+\infty}\left(1+\dfrac{e}{x}\right)^{x/2}=\lim\limits_{y\to0^+}\left(1+ey\right)^{1/2y}$. Next, let $z=\left(1+ey\right)^{1/2y}\Rightarrow \ln z=$

$$\ln\left(1+ey\right)^{1/2y}=\frac{\ln(1+ey)}{2y}\Rightarrow \lim_{y\to0^+}\ln z=\lim_{y\to0^+}\frac{\ln(1+ey)}{2y}=\lim_{y\to0^+}\frac{\frac{e}{1+ey}}{2}=\frac{e}{2}.$$ Thus,

$$\lim_{y\to 0^+}\ln z=\frac{e}{2}\Rightarrow \ln\left(\lim_{y\to 0^+} z\right)=\frac{e}{2}\Rightarrow \lim_{y\to 0^+} z=e^{e/2}\Rightarrow \lim_{x\to+\infty}\left(1+\frac{e}{x}\right)^{x/2}=\lim_{y\to 0^+}(1+ey)^{1/2y}=$$

$$\lim_{y\to 0^+} z=e^{e/2}\,.\; y=\left(e^{2x}+x\right)^{1/x}\Rightarrow \ln y=\ln\left(e^{2x}+x\right)^{1/x}=\frac{\ln\left(e^{2x}+x\right)}{x}\Rightarrow \lim_{x\to 0}\ln y=$$

(18) $\displaystyle \lim_{x\to 0}\frac{\arctan(\sin 3x)}{x}=\lim_{x\to 0}\frac{\frac{3\cos 3x}{1+\sin^2 3x}}{1}=3\,.$

(19) $\displaystyle \lim_{x\to 0^+}\frac{\int_0^x \sin\left(t^2\right)dt}{x^3}=\lim_{x\to 0^+}\frac{\sin(x^2)}{3x^2}=\lim_{x\to 0^+}\frac{2x\cos(x^2)}{6x}=\lim_{x\to 0^+}\frac{\cos(x^2)}{3}=\frac{1}{3}\,.$

(20) Let $y=\left(e^{2x}+x\right)^{1/x}\Rightarrow \ln y=\ln\left(e^{2x}+x\right)^{1/x}=\dfrac{\ln\left(e^{2x}+x\right)}{x}\Rightarrow \lim_{x\to 0}\ln y=$

$\displaystyle \lim_{x\to 0}\frac{\ln\left(e^{2x}+x\right)}{x}=\lim_{x\to 0}\frac{\frac{2e^{2x}+1}{e^{2x}+x}}{1}=3$. Thus $\displaystyle \lim_{x\to 0}\ln y=3\Rightarrow \ln\left(\lim_{x\to 0} y\right)=3\Rightarrow$

$\displaystyle \lim_{x\to 0} y=e^3\Rightarrow \lim_{x\to 0}\left(e^{2x}+x\right)^{1/x}=\lim_{x\to 0} y=e^3\,.$

(21) $\displaystyle \lim_{x\to+\infty}\frac{\arctan x}{x}=\frac{\pi/2}{+\infty}=0\,.$

(22) $\displaystyle \lim_{x\to 0}\frac{\arctan(\sin(3x))}{\arcsin(2\tan x)}=\lim_{x\to 0}\frac{\frac{3\cos 3x}{1+\sin^2 3x}}{\frac{2\sec^2 x}{\sqrt{1-4\tan^2 x}}}=\frac{3/1}{2/1}=\frac{3}{2}\,.$

(23) $\displaystyle \lim_{x\to 0}\frac{\ln(\cos x)}{x^2}=\lim_{x\to 0}\frac{\frac{-\sin x}{\cos x}}{2x}=\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\left(\frac{-1}{2\cos x}\right)=-\frac{1}{2}\,.$

(24) Let $\displaystyle y=\frac{1}{x}\Rightarrow \lim_{x\to+\infty}\left(1+\frac{1}{2x}\right)^{x}=\lim_{y\to 0^+}\left(1+\frac{1}{2}y\right)^{1/y}$. Let $\displaystyle z=\left(1+\frac{1}{2}y\right)^{1/y}\Rightarrow \ln z=$

$$\ln\left(1+\frac{1}{2}y\right)^{1/y} = \frac{\ln\left(1+\frac{1}{2}y\right)}{y} \Rightarrow \lim_{y\to 0^+}\ln z = \lim_{y\to 0^+}\frac{\ln\left(1+\frac{1}{2}y\right)}{y} = \lim_{y\to 0^+}\frac{\frac{1/2}{1+1/2\,y}}{1} = \frac{1}{2}.$$

Thus, $\lim_{y\to 0^+}\ln z = \frac{1}{2} \Rightarrow \ln\left(\lim_{y\to 0^+} z\right) = \frac{1}{2} \Rightarrow \lim_{y\to 0^+} z = e^{1/2} \Rightarrow \lim_{x\to +\infty}\left(1+\frac{1}{2x}\right)^x =$

$\lim_{y\to 0^+}\left(1+\frac{1}{2}y\right)^{1/y} = \lim_{y\to 0^+} z = e^{1/2}.$

(25) Let $y = (\ln x)^{1/x} \Rightarrow \ln y = \ln(\ln x)^{1/x} = \frac{\ln(\ln x)}{x} \Rightarrow \lim_{x\to +\infty}\ln y = \lim_{x\to +\infty}\frac{\ln(\ln x)}{x} =$

$\lim_{x\to +\infty}\frac{\frac{1}{x\ln x}}{1} = \lim_{x\to +\infty}\frac{1}{x\ln x} = 0$. Thus, $\lim_{x\to +\infty}\ln y = 0 \Rightarrow \ln\left(\lim_{x\to +\infty} y\right) = 0 \Rightarrow \lim_{x\to +\infty} y =$

$e^0 = 1 \Rightarrow \lim_{x\to +\infty}(\ln x)^{1/x} = \lim_{x\to +\infty} y = 1$.

(26) $\lim_{x\to +\infty}\left(\ln(e^x+1) - x\right) = \lim_{x\to +\infty}\left(\ln\left(e^x+1\right) - \ln e^x\right) = \lim_{x\to +\infty}\left(\ln\left(\frac{e^x+1}{e^x}\right)\right) =$

$\ln\left(\lim_{x\to +\infty}\left(\frac{e^x+1}{e^x}\right)\right) = \ln\left(\lim_{x\to +\infty}\left(\frac{e^x}{e^x}\right)\right) = \ln 1 = 0$.